S4A1 - SEMINAR ON ALGEBRAIC GEOMETRY
JACOBIANS OF CURVES

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The aim of the seminar is to prove the following theorem.

Theorem 1. *Let k be an algebraically closed field and $f\colon X \to \mathrm{Spec}(k)$ a projective, smooth, irreducible curve. Fixing some $x \in X(k)$, define the Picard functor of X over k by*

$$\mathrm{Pic}_{X/k,x}\colon (\mathrm{Sch}/k)^{\mathrm{op}} \to (\mathrm{Sets}),\ T \mapsto \mathrm{Ker}(\mathrm{Pic}(X_T) \xrightarrow{s_T^*} \mathrm{Pic}(T)),$$

where X_T resp. s_T denote the following base changes,

$$X_T := X \times_{\mathrm{Spec}(k)} T \quad \textit{resp.} \quad s_T = x \times \mathrm{Id}_T.$$

This functor $\mathrm{Pic}_{X/k,x}$ is representable by a scheme.

Moreover, we will analyze the representing scheme of $\mathrm{Pic}_{X/k}$ more closely. For example, we will see that it is a smooth k-scheme of dimension equal to the genus of X and that its connected components are proper. Moreover, the $\otimes$-product of line bundles defines a group structure on $\mathrm{Pic}_{X/k}$. The connected component of the identity element is a so-called abelian variety and we will study such group schemes more generally in the second half of the seminar.

We now outline the proof of Theorem 1 and indicate the contents of the talks.

Let X and k be as above. The idea is to consider the map that sends an effective Cartier divisor D on X to its line bundle $\mathcal{O}_X(D)$ (Talk 1), the so-called Abel–Jacobi map

$$\mathrm{AJ}'\colon \{\text{effective Cartier divisors on } X \text{ of degree d}\} \to \mathrm{Pic}^d(X).$$

The domain of AJ' is actually the set of k-points $\mathrm{Hilb}^d_{X/k}(k)$ of the so-called Hilbert scheme of points (Talks 3, 4),

$$\mathrm{Hilb}^d_{X/k}.$$

It parametrizes divisors of degree d on X and admits a very concrete description. Namely, an effective Cartier divisor of degree d on X is the same as an unordered tuple $(x_1, \dots, x_d)$ of points $x_1, \dots, x_d \in X(k)$. Thus we expect that the Hilbert scheme of points $\mathrm{Hilb}^d_{X/k}$ is isomorphic to the symmetric product

$$X^{(d)} := X^d/S_d \cong \mathrm{Hilb}^d_{X/k},$$

where the symmetric group S_d acts on the product X^d by permuting the factors. This intuition can be made precise (Talk 5). We obtain a scheme version of the Abel–Jacobi map,

$$\mathrm{AJ}\colon \mathrm{Hilb}^d_{X/k} \to \mathrm{Pic}_{X/k,x},$$

which at this point is merely a natural transformation of functors. In order to deduce the representability of the Picard functor $\mathrm{Pic}_{X/k}$ (Talks 6, 7) we will analyze the geometry of the map AJ more closely. Namely, we will show that if g is the genus of X, then there exists an open subset $V \subseteq \mathrm{Hilb}^g_{X/\mathrm{Spec}(k)}$ which maps via AJ isomorphically onto an open subfunctor of $\mathrm{Pic}^g_{X/k}$. Using

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the group structure of $\mathrm{Pic}_{X/k}$, we may translate this open subfunctor $\mathrm{AJ}(V)$ around to produce a covering

$$\mathrm{Pic_{X/k}} = \bigcup_{\mathcal{L}\in \mathrm{Pic(X)}} \mathcal{L} + \mathrm{AJ}(V)$$

by representable open subfunctors, thus establishing the representability of $\mathrm{Pic}_{X/k}$ itself.

With Theorem 1 in hand, we will study the Abel–Jacobi map more closely (Talk 8) and see that its fibers are given by projective spaces. We will also explore variants of Theorem 1 for singular curves (Talk 9) and Riemann surfaces (Talk 10).

Coming back to the Picard functor, it is seen to decompose into a disjoint union

$$\mathrm{Pic}_{X/k} \cong \coprod_{d\in\mathbb{Z}} \mathrm{Pic}^d_{X/k}$$

with the property that $\mathcal{L} \in \mathrm{Pic}(X)$ lies in $\mathrm{Pic}^d_{X/k}(k)$ precisely for $d = \deg \mathcal{L}$. The scheme $\mathrm{Pic}^0_{X/k}$, which thus parametrizes line bundles of degree 0 on X, is called the Jacobian of X and is the basic example of an abelian variety k, meaning it is a proper, smooth and connected group scheme over k.

With this example in mind we discuss abelian varietes a bit further. In particular, we prove that abelian varieties are "abelian", i.e. that their group structure is necessarily commutative (Talk 11). We also determine their torsion (Talk 12). Finally, we will define and construct the dual abelian variety (Talk 14) which also requires us to study quotients of abelian varieties by finite subgroups more closely (Talk 13). Jacobians of curves turn out to be of a special kind, namely they are always self-dual via the Θ-divisor.

Talks

Two general rules apply. First, every talk should stay within its time frame and it is the task of the speaker to arrange the material accordingly. Second, every speaker should contact the organizers at least one week before his/her talk to clarify its content.

Talk 1 - Line bundles on curves (05.04.22)
Define effective Cartier divisors [Stacks, Tag 01WR] on general schemes and explain their relation with line bundles [Stacks, Tag 01X0]. Then move on to divisors on normal curves over fields and present [GW10, Section (15.9), Section (15.10)]. (You may assume curves to be normal here). State [Liu02, Proposition 7.5.] that a divisor on a projective irreducible curve is ample if and only if its degree is positive (in particular, define ampleness [GW10, Definition 13.44]). If time permits, present the singular example [Har77, Example II.6.11.4].

Talk 2 - Smooth morphisms (12.04.22)
Present [GW10, Section (6.8), Section (6.10)], [GW10, Theorem 6.28], its corollary [GW10, Corollary 6.34], [GW10, Example 6.34] and [GW10, Theorem 14.22]. Then explain as much as possible of [Stacks, Tag 01V8] (or [BLR90, Proposition 2.2/8] which might be more accessible).

Talk 3 - The Hilbert scheme of points (19.04.22)
Follow [Stacks, Chapter 44, Section 2] and present the Tags 0B96-0B99, 0B9A, 0B9B.

Talk 4 - The Hilbert scheme of points for relative curves (26.04.22)
Present [Stacks, Tags 0B9C-0B9J], i.e., [Stacks, Chapter 44, Section 3]. In particular, define relative effective Cartier divisors [Stacks, Tag 062T].

Talk 5 - Symmetric powers (03.05.22)
Present [GW10, Proposition 12.27] resp. [Gro71, Exposé V.1, Proposition 1.1., Proposition 1.8.], deduce the existence of symmetric powers $(X/S)^{(n)}$ for quasi-projective morphisms $f\colon X \to S$ and prove [BLR90, Chapter 9.3., Proposition 3]. Finally, discuss the examples $(\mathbb{A}^1_S/S)^{(n)} \cong \mathbb{A}^n_S$ and $(\mathbb{P}^1_S/S)^{(n)} \cong \mathbb{P}^n_S$.

Talk 6 - Representability of the Picard functor I (10.05.22)
Present [Stacks, Tag 0B9Q] in full detail, in particular recall [Stacks, Tag 01JJ]. Then present [Stacks, Tags 0B9V, 0B9W, 0B9X] (with a more detailed discussion of the cohomological results that are cited there).

Talk 7 - Representability of the Picard functor II (17.05.22)
Present the rest of [Stacks, Chapter 44, Section 6], i.e., Tags 0B9Y, 0B9Z, 0BA0. Finally discuss the cases $g = 0, 1$. Deduce that elliptic curves are canonically equipped with a group structure.

Talk 8 - The Abel–Jacobi map in general (24.05.22)
Recall the notion of relative representability [Stacks, 0023]. Then present [BLR90, Proposition 8.2/7] with all necessary preliminaries. This includes [BLR90, Lemma 8.2/6] and [BLR90, Theorem 8.1/7].

Talk 9 - The Picard functor for singular curves (31.05.22)
Give [BLR90, 8.2/3 and 8.4/2], stating that $\mathrm{Pic}_{X/k}$ is representable by a smooth scheme also for singular X. Then present the material in [BLR90, Chapter 9.2] from Example 8 until Corollary 12. Assume throughout that K is algebraically closed and X reduced, skip Corollary 11.

Talk 10 - Jacobians of compact Riemann surfaces (07.06.22)
For this talk some knowledge of complex analysis and deRham cohomology is helpful. Present [Ara, Sections 2-4].

Talk 11 - Abelian varieties I (14.06.22)
Define abelian varieties [Mum70, II.4] and prove that their group structure is abelian [Mum70, II.4, Corollary 2]. State (or prove if possible) the Seesaw Theorem [Mum70, II.5, Corollary 6], the theorem of the cube [Mum70, II.6] and deduce the corollaries 1–4 in [Mum70, II.6].

Talk 12 - Abelian varietes II (21.06.22)
Present the rest of [Mum70, II.6], i.e., start with Application 1. You can omit one of the proofs that $\deg(n_X) = n^{2g}$.

Talk 13 - Separable isogenies of abelian varieties (28.06.22)
Present [Mum70, II.7]. Note that some parts of the main theorem have been covered in Talk 5. For the proof of [Mum70, II.7, Proposition 2] you may also cite results on faithfully flat descent if they have been covered in the lecture. If you are very ambitious you may discuss the case of inseparable isogenies as well, see [Mum70, II.12].

Talk 14 - The dual abelian variety in characteristic 0 (05.07.22)
Present the material in [Mum70, II.8] until the end of the proof of Theorem 1. End your talk and the seminar by defining the Θ-divisor and giving its properties, cf. [Mil, Theorem 6.6].

References

[Ara] D. Arapura, *The Jacobian of a Riemann surface* , Preprint, http://www.math.purdue.edu/ dvb/preprints/jacobian.pdf.

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[GW10] U. Görtz and T. Wedhorn, *Algebraic Geometry I: Schemes*, Springer Spektrum, Wiesbaden, 2010.

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[Mil] J. S. Milne, *Jacobian Varieties*, In Arithmetic geometry, edited by Cornell G. and Silverman J. H., pages 167–212. Springer, New York, 1986.

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